

Adler Function, Sum Rules and Crewther Relation of Order $O(\alpha_s^4)$: the Singlet Case

P. A. Baikov^a, K. G. Chetyrkin^b, J. H. Kühn^b, J. Rittinger^b

^a Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, 1(2), Leninskie gory, Moscow 119234, Russian Federation

^b Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), Wolfgang-Gaede-Straße 1, 726128 Karlsruhe, Germany

Abstract

The analytic result for the singlet part of the Adler function of the vector current in a general gauge theory is presented in five-loop approximation. Comparing this result with the corresponding singlet part of the Gross-Llewellyn Smith sum rule [1], we successfully demonstrate the validity of the generalized Crewther relation for the singlet part. This provides a non-trivial test of both our calculations and the generalized Crewther relation. Combining the result with the already available non-singlet part of the Adler function [2, 3] we arrive at the complete $O(\alpha_s^4)$ expression for the Adler function and, as a direct consequence, at the complete $O(\alpha_s^4)$ correction to the e^+e^- annihilation into hadrons in a general gauge theory.

Key words: QCD, Adler function, Gross-Llewellyn Smith sum rule, Crewther relation

1. Introduction

The classical Crewther relation (CR) [4] connects in a non-trivial way two seemingly unrelated quantities, namely the Adler function [5] and perturbative corrections arising in the sum rules relevant for deep inelastic scattering (DIS). Originally the CR had been formulated for the case of a conformal-invariant limit of a field theory. Subsequently, observing a close relation between the $O(\alpha_s^3)$ terms in the Adler function and the corrections to the Bjorken sum rule for polarized electron-nucleon scattering, its generalization for the case of QCD was suggested in [6], introducing as modification additional terms proportional to the *beta*-function. More formal arguments for the validity of this “generalized Crewther relation” (GCR) were given in [7, 8, 9]. During the past years the perturbative corrections both for Adler function and Bjorken sum rule were extended from $O(\alpha_s^3)$ [10, 11, 12, 13] to $O(\alpha_s^4)$ [2, 3]. However, these results were restricted to the respective non-singlet parts. Nevertheless, they could be used to demonstrate the validity of the GCR between non-singlet Adler function and Bjorken sum rule [14, 15], thus providing at the same time an important cross check of the underlying, demanding calculations.

The $O(\alpha_s^4)$ singlet piece of the sum rule was published in [1], thus completing the prediction for the Gross-Llewellyn Smith (GLS) sum rule [16]. Below we give the corresponding result for the Adler function. On the one hand this leads to a prediction of the familiar *R*-ratio measured in electron-positron annihilation, including the (small) up to now missing singlet pieces of $O(\alpha_s^4)$, on the other hand this result allows to test the GCR also for the singlet case. Note that all results discussed in this paper are assumed to be renormalized within the conventional $\overline{\text{MS}}$ subtraction scheme [17].

2. Singlet $O(\alpha_s^4)$ contributions to the Adler function and $R(s)$

For the definition of the Adler function it is convenient to start with the polarization function of the flavor singlet vector

current:

$$3 Q^2 \Pi(Q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T j_\mu(x) j^\mu(0) | 0 \rangle, \quad (1)$$

with $j_\mu = \sum_i \bar{\psi}_i \gamma_\mu \psi_i$ and $Q^2 = -q^2$. The corresponding Adler function

$$D(Q^2) = -12 \pi^2 Q^2 \frac{d}{dQ^2} \Pi(Q^2) \quad (2)$$

is naturally decomposed into a sum of the non-singlet (NS) and singlet (SI) components (see Fig. 1):

$$D(Q^2) = n_f D^{NS}(Q^2) + n_f^2 D^{SI}(Q^2). \quad (3)$$

Here n_f stands for the total number of quark flavours; all quarks are considered as massless. The Adler function D^{EM} corresponding to the electromagnetic vector current $j_\mu^{EM} = \sum_i q_i \bar{\psi}_i \gamma_\mu \psi_i$ (q_i stands for the electric charge of the quark field ψ_i) is thus given by the following combination:

$$D^{EM} = \left(\sum_i q_i^2 \right) D^{NS} + \left(\sum_i q_i \right)^2 D^{SI}. \quad (4)$$

Similar decompositions hold for the corresponding polarization functions Π and Π^{EM} .

The physical observable $R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$ is related to $\Pi^{EM}(Q^2)$ by the optical theorem

$$R(s) = 12\pi \Im \Pi^{EM}(-s - i\epsilon). \quad (5)$$

The result for the perturbative expansions of the non-singlet part ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$D^{NS}(Q^2) = d_R \left(1 + \sum_{i=1}^{\infty} d_i^{NS} a_s^i(Q^2) \right) \quad (6)$$

has been presented in [3]. For the singlet part it reads:

$$D^{SI}(Q^2) = d_R \left(\sum_{i=3}^{\infty} d_i^{SI} a_s^i(Q^2) \right), \quad (7)$$

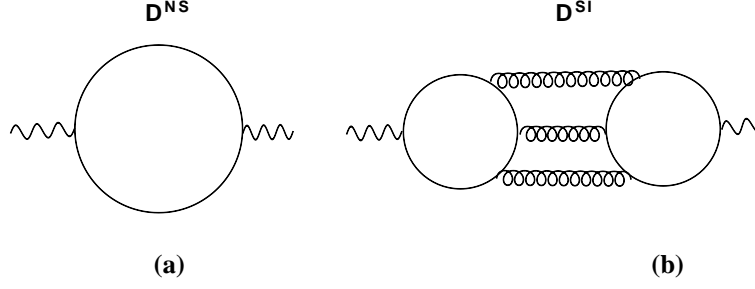


Figure 1: Lowest order non-singlet (a) and singlet (b) diagrams contributing to the Adler function.

where the parameter d_R (the dimension of the quark color representation, $d_R = 3$ in QCD) is factorized in *both* non-singlet and singlet components.

The singlet component has the following structure at orders α_s^3 and α_s^4 :

$$d_3^{SI} = d^{abc} d^{abc} / d_R \left(\frac{11}{192} - \frac{1}{8} \zeta_3 \right), \quad (8)$$

$$d_4^{SI} = d^{abc} d^{abc} / d_R \left(C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T n_f d_{4,3}^{SI} \right). \quad (9)$$

Here C_F and C_A are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra, $d^{abc} = 2 \text{Tr}(\{\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\}, \frac{\lambda^c}{2})$, T is the trace normalization of the fundamental representation. For QCD (colour gauge group $SU(3)$):

$$C_F = 4/3, \quad C_A = 3, \quad T = 1/2, \quad d^{abc} d^{abc} = 40/3.$$

Using the methods described in [18, 19, 20, 2, 21, 3] we obtain

$$d_{4,1}^{SI} = -\frac{13}{64} - \frac{1}{4} \zeta_3 + \frac{5}{8} \zeta_5, \quad (10)$$

$$d_{4,2}^{SI} = \frac{3893}{4608} - \frac{169}{128} \zeta_3 + \frac{45}{64} \zeta_5 - \frac{11}{32} \zeta_3^2, \quad (11)$$

$$d_{4,3}^{SI} = -\frac{149}{576} + \frac{13}{32} \zeta_3 - \frac{5}{16} \zeta_5 + \frac{1}{8} \zeta_3^2. \quad (12)$$

With the use of eqs. (10)-(12) and the result for D^{NS} from [3] we arrive at the complete result for the ratio $R(s)$ at order α_s^4 in (massless) QCD:

$$\begin{aligned} R(s) = 3 \sum_f q_f^2 & \left\{ 1 + a_s + a_s^2 \left(\frac{365}{24} - 11 \zeta_3 - \frac{11}{12} n_f + \frac{2}{3} \zeta_3 n_f \right) \right. \\ & + a_s^3 \left[n_f^2 \left(\frac{151}{162} - \frac{1}{108} \pi^2 - \frac{19}{27} \zeta_3 \right) \right. \\ & + n_f \left(-\frac{7847}{216} + \frac{11}{36} \pi^2 + \frac{262}{9} \zeta_3 - \frac{25}{9} \zeta_5 \right) \\ & + \frac{87029}{288} - \frac{121}{48} \pi^2 - \frac{1103}{4} \zeta_3 + \frac{275}{6} \zeta_5 \left. \right] \\ & + a_s^4 \left[n_f^3 \left(-\frac{6131}{5832} + \frac{11}{432} \pi^2 + \frac{203}{324} \zeta_3 - \frac{1}{54} \pi^2 \zeta_3 + \frac{5}{18} \zeta_5 \right) \right. \\ & + n_f^2 \left(\frac{1045381}{15552} - \frac{593}{432} \pi^2 - \frac{40655}{864} \zeta_3 \right. \\ & + \frac{11}{12} \pi^2 \zeta_3 + \frac{5}{6} \zeta_3^2 - \frac{260}{27} \zeta_5 \left. \right) \\ & + n_f \left(-\frac{13044007}{10368} + \frac{2263}{96} \pi^2 + \frac{12205}{12} \zeta_3 - \frac{121}{8} \pi^2 \zeta_3 \right. \\ & - 55 \zeta_3^2 + \frac{29675}{432} \zeta_5 + \frac{665}{72} \zeta_7 \left. \right) \\ & + \frac{144939499}{20736} - \frac{49775}{384} \pi^2 - \frac{5693495}{864} \zeta_3 + \frac{1331}{16} \pi^2 \zeta_3 \\ & \left. \left. + \frac{5445}{8} \zeta_3^2 + \frac{65945}{288} \zeta_5 - \frac{7315}{48} \zeta_7 \right] \right\} \end{aligned}$$

$$\begin{aligned} & + \left(\sum_f q_f \right)^2 \left\{ a_s^3 \left(\frac{55}{72} - \frac{5}{3} \zeta_3 \right) \right. \\ & + a_s^4 \left[n_f \left(-\frac{745}{432} + \frac{65}{24} \zeta_3 + \frac{5}{6} \zeta_3^2 - \frac{25}{12} \zeta_5 \right) \right. \\ & \left. \left. + \left(\frac{5795}{192} - \frac{8245}{144} \zeta_3 - \frac{55}{4} \zeta_3^2 + \frac{2825}{72} \zeta_5 \right) \right] \right\}, \quad (13) \end{aligned}$$

where we set $\mu = Q$. The full results for Adler function and $R(s)$ for generic color factors and generic value of μ are rather lengthy and can be found available (in computer-readable form) in <http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp12/t>. Numerically, it reads:

$$\begin{aligned} R(s) = 3 \sum_f q_f^2 & \left\{ 1 + a_s + a_s^2 (1.986 - 0.1153 n_f) \right. \\ & + a_s^3 (-6.637 - 1.200 n_f - 0.00518 n_f^2) \\ & + a_s^4 (-156.608 + 18.7748 n_f - 0.797434 n_f^2 \\ & \left. + 0.0215161 n_f^3) \right\} \\ & - \left(\sum_f q_f \right)^2 (1.2395 a_s^3 + (17.8277 - 0.57489 n_f) a_s^4). \end{aligned}$$

Specifically, for the particular values of $n_f = 4$ and 5 one obtains (for the terms of order α_s^3 and α_s^4 we have explicitly decomposed the coefficient into non-singlet and singlet contributions):

$$\begin{aligned} R^{n_f=4}(s) = \frac{10}{3} & \left[1 + a_s + 1.5245 a_s^2 \right. \\ & + a_s^3 (-11.686 = -11.52 - 0.16527^{SI}) \\ & \left. + a_s^4 (-94.961 = -92.891 - 2.0703^{SI}) \right], \quad (14) \end{aligned}$$

$$\begin{aligned} R^{n_f=5}(s) = \frac{11}{3} & \left[1 + a_s + 1.40902 a_s^2 \right. \\ & + a_s^3 (-12.80 = -12.767 - 0.037562^{SI}) \\ & \left. + a_s^4 (-80.434 = -79.981 - 0.4531^{SI}) \right]. \quad (15) \end{aligned}$$

Note that for $n_f = 3$ the singlet contributions vanish in every order in α_s as the corresponding global coefficient $(\sum_i q_i)^2$ happens to be zero. Implications of this result for the determination of α_s in electron-positron annihilation and in Z-boson decays are discussed in [22].

3. GLS sum rule at order $O(\alpha_s^4)$ and the Crewther relation

The second quantity of interest, the GLS sum rule,

$$\frac{1}{2} \int_0^1 F_3(x, Q^2) dx = 3 C^{CLS}(a_s), \quad (16)$$

relates the lowest moment of the isospin singlet structure function $F_3^{\nu p+\bar{\nu} p}(x, Q^2)$ to a coefficient $C^{CLS}(a_s)$, which appears in the operator product expansion of the axial and vector non-singlet currents

$$i \int T A_\mu^a(x) V_\nu^b(0) e^{iqx} dx|_{q^2 \rightarrow -\infty} \approx C_{\mu\nu\alpha}^{V,ab} V_\alpha(0) + \dots \quad (17)$$

where

$$C_{\mu\nu\alpha}^{V,ab} = \delta^{ab} \epsilon_{\mu\nu\alpha\beta} \frac{q^\beta}{Q^2} C^{GLS}(a_s)$$

and $V_\alpha = \bar{\psi} \gamma_\alpha \psi$ is a flavour singlet quark current. At last $A_\mu^a = \bar{\psi} \gamma_\mu \gamma_5 t^a \psi$, $V_\nu^b = \bar{\psi} \gamma_\nu t^b \psi$ are axial vector and vector non-singlet quark currents, with t^a , t^b being the generators of the flavour group $SU(n_f)$.

Again diagrams contributing to $C^{GLS}(a_s)$ can be separated in two groups: non-singlet and singlet ones (see Fig. 2):

$$C^{GLS} = C^{NS} + C^{SI}, \quad (18)$$

$$C^{NS}(Q^2) = 1 + \sum_{i=1}^{\infty} c_i^{NS} a_s^i(Q^2), \quad (19)$$

$$C^{SI}(Q^2) = \sum_{i=3}^{\infty} c_i^{SI} a_s^i(Q^2). \quad (20)$$

The results for both functions C^{NS} and C^{SI} at order α_s^3 are known since early 90-ties [13]. Note that as a consequence of chiral invariance the closely related Bjorken sum rule receives contributions from the non-singlet piece only [13]:

$$C^{Bjp} \equiv C^{NS}. \quad (21)$$

The $O(\alpha_s^4)$ contribution to C^{Bjp} has been computed some time ago [3]. The calculation of the $O(\alpha_s^4)$ contribution to C^{SI} has been published in [1] for a generic gauge group and is repeated below:

$$c_3^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} \left(c_{3,1}^{SI} \equiv -\frac{11}{192} + \frac{1}{8} \zeta_3 \right), \quad (22)$$

$$c_4^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} \left(C_F c_{4,1}^{SI} + C_A c_{4,2}^{SI} + T n_f c_{4,3}^{SI} \right), \quad (23)$$

$$c_{4,1}^{SI} = \frac{37}{128} + \frac{1}{16} \zeta_3 - \frac{5}{8} \zeta_5, \quad (24)$$

$$c_{4,2}^{SI} = -\frac{481}{1152} + \frac{971}{1152} \zeta_3 - \frac{295}{576} \zeta_5 + \frac{11}{32} \zeta_3^2, \quad (25)$$

$$c_{4,3}^{SI} = \frac{119}{1152} - \frac{67}{288} \zeta_3 + \frac{35}{144} \zeta_5 - \frac{1}{8} \zeta_3^2. \quad (26)$$

Using the input from eqs. (10-12) and (22-26), the validity of the GCR can now be investigated. In fact, there exist two of

them [23, 6], one involving the non-singlet parts only and one involving also a singlet piece:

$$\begin{aligned} D^{NS}(a_s) C^{Bjp}(a_s) &= d_R \left[1 + \frac{\beta(a_s)}{a_s} K^{NS}(a_s) \right], \\ K^{NS}(a_s) &= a_s K_1^{NS} + a_s^2 K_2^{NS} + a_s^3 K_3^{NS} + \dots \end{aligned} \quad (27)$$

and

$$\begin{aligned} D(a_s) C^{GLS}(a_s) &= d_R n_f \left[1 + \frac{\beta(a_s)}{a_s} K(a_s) \right], \\ K(a_s) &= a_s K_1 + a_s^2 K_2 + a_s^3 K_3 + \dots \end{aligned} \quad (28)$$

Here $\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i \geq 0} \beta_i a_s^{i+2}$ is the QCD β -function with its first term $\beta_0 = \frac{11}{12} C_A - \frac{T}{3} n_f$. The term proportional to the β -function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order α_s^2 .

Relation (27) has been studied in detail in [3], where its validity at order α_s^4 was demonstrated (a detailed discussion at orders α_s^2 and α_s^3 can be found in [6]).

Let us consider now eq. (28). Combining eqs. (3,18,21) and (27) leads to the following relations between coefficients K_i^{NS} and K_i :

$$K_1 = K_1^{NS}, \quad K_2 = K_2^{NS}, \quad (29)$$

$$K_3 = K_3^{NS} + K_3^{SI}, \quad (30)$$

$$K_3^{SI} = k_{3,1}^{SI} n_f \frac{d^{abc} d^{abc}}{d_R}, \quad (31)$$

with $k_{3,1}^{SI}$ being a numerical parameter.

Thus, we conclude that eq. (28) puts $3 - 1 = 2$ constraints between two triplets of (purely numerical) parameters $\{d_{4,1}^{SI}, d_{4,2}^{SI}, d_{4,3}^{SI}\}$ and $\{c_{4,1}^{SI}, c_{4,2}^{SI}, c_{4,3}^{SI}\}$ appearing in eqs. (9) and (23) and completely describing the order α_s^4 singlet contributions to the Adler function and the Gross-Llewellyn Smith sum rule respectively.

The solution of the constraints and eqs. (24-26) produces the following relations for $d_{4,i}^{SI}$:

$$d_{4,1}^{SI} = -\frac{3}{2} c_{3,1}^{SI} - c_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad (32)$$

$$d_{4,2}^{SI} = -c_{4,2}^{SI} - \frac{11}{12} k_{3,1}^{SI}, \quad (33)$$

$$d_{4,3}^{SI} = -c_{4,3}^{SI} + \frac{1}{3} k_{3,1}^{SI}, \quad (34)$$

whose validity is indeed confirmed by the explicit calculations. As a result the remaining unknown $k_{3,1}^{SI}$ is fixed as:

$$k_{3,1}^{SI} = -\frac{179}{384} + \frac{25}{48} \zeta_3 - \frac{5}{24} \zeta_5. \quad (35)$$

4. Conclusion

We have analytically computed coefficients of all three colour structures contributing to the singlet part of the Adler function in massless QCD at $O(\alpha_s^4)$. We have checked that all constraints on these coefficients derived previously in [1] on the base of the

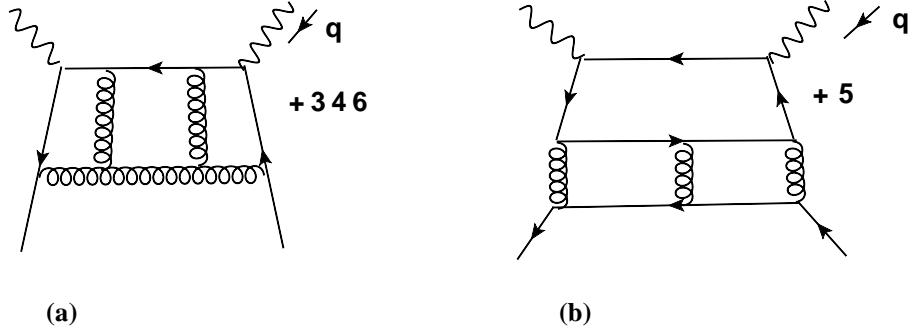


Figure 2: (a),(b): $O(\alpha_s^3)$ non-singlet and singlet diagrams contributing to the Gross-Llewellyn Smith sum rule; note that the coefficient function C^{Bjp} is contributed by only non-singlet diagrams.

GCR are really fulfilled. This is an important cross-check of our calculations of D^{SI} , C^{SI} and the very GCR.

The calculations has been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using parallel MPI-based [24] as well as thread-based [25] versions of FORM [26]. For the evaluation of color factors we have used the FORM program *COLOR* [27]. The diagrams have been generated with QGRAF [28]. The figures have been drawn with the the help of Axodraw [29] and JaxoDraw [30].

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